

Electromagnetism: Electrostatics

FIZIKA SPhO Training

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1 Notes

In electrostatics, we analyse the forces and motion of static charges. Hereby, we shall take $k = \frac{1}{4\pi\epsilon_0} \approx 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$.

1.1 The Basic Quantities

In electrostatics, the four basic quantities you will deal with are:

1. Electric Force, \mathbf{F}_E
2. Electric Field, \mathbf{E}
3. Electric Potential Energy, U_E
4. Electric Potential, V_E

Notice that these quantities are *very similar* to what you've seen in gravitation!

1.1.1 Electric Force and Electric Field

Consider two point charges q_1 and q_2 located at position vectors \mathbf{r}_1 and \mathbf{r}_2 . Let the separation vector be $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

The electric forces by each point charge on the other charge are given by

$$\mathbf{F}_{E,1 \text{ by } 2} = -\frac{kq_1q_2}{|\mathbf{r}|^3} \mathbf{r} \quad (1)$$

$$\mathbf{F}_{E,2 \text{ by } 1} = \frac{kq_1q_2}{|\mathbf{r}|^3} \mathbf{r} \quad (2)$$

The electric field can be thought of as the electric force per unit charge. Let the charge Q be placed at the origin. The electric field at a position vector \mathbf{r} away is given by

$$\mathbf{E} = \frac{kQ}{|\mathbf{r}|^3} \mathbf{r} \quad (3)$$

Remark. The signs of the charges in Equations (1) to (3) are important!

1.1.2 Electric Potential Energy and Electric Potential

Consider two point charges again. The electric potential energy is given by

$$U_E = \frac{kq_1q_2}{|\mathbf{r}|} \quad (4)$$

The electric potential can be thought of as the electric potential energy per unit charge. Thus, the electric potential at a position vector \mathbf{r} away is given by

$$V_E = \frac{kQ}{|\mathbf{r}|} \quad (5)$$

Notice that Equations (1) to (5) look *very similar* to Equations (1) to (5) in the gravitation handout! We can draw an analogy between electrostatics and gravitation:

$$k \longleftrightarrow G, \quad q \longleftrightarrow m \quad (6)$$

which means any electrostatics problem can be solved as a gravitation one, and vice-versa!

1.1.3 Relationships Between Quantities

You should expect similar relationships to hold, as in gravitation:

$$F_E = -\frac{dU_E}{dr} \quad (7)$$

$$U_E = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_E \cdot d\mathbf{r} \quad (8)$$

$$E = -\frac{dV_E}{dr} \quad (9)$$

$$V_E = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r} \quad (10)$$

The "square" between the four quantities you saw in the gravitation handout can also be used.

1.2 Continuous Charge Distributions

We have been talking about discrete charge distributions, but what about continuous charge distributions? Continuous charge distributions can have either linear charge density λ , surface charge density σ , or volume charge density ρ .

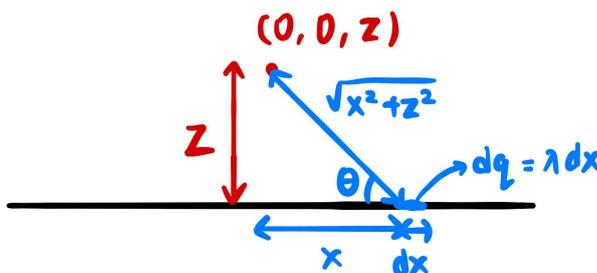
1.2.1 Naive Integration

We are usually interested in finding the electric field or electric potential. One way to attack such problems is through naive integration, as the example below illustrates.

Example 1.1. Find the electric field at a distance z above the midpoint of a straight wire of length $2L$ that carries a uniform linear charge density λ .

Let the wire lie along the x -axis with its centre at the origin. Clearly, due to symmetry, only the z -component of the electric field survives upon integration across the wire.

Consider an infinitesimal charge $dq = \lambda dx$ in the figure below:



The infinitesimal electric field produced by this infinitesimal charge is

$$dE = \frac{k\lambda dx}{r^2} = \frac{k\lambda dx}{x^2 + z^2}$$

The net electric field is the sum of all the z -components, hence

$$E_{net} = \int dE \sin \theta = \int_{-L}^L \frac{k\lambda dx}{x^2 + z^2} \frac{z}{\sqrt{x^2 + z^2}} = k\lambda z \int_{-L}^L \frac{dx}{(x^2 + z^2)^{\frac{3}{2}}} = \frac{2k\lambda L}{z\sqrt{L^2 + z^2}}$$

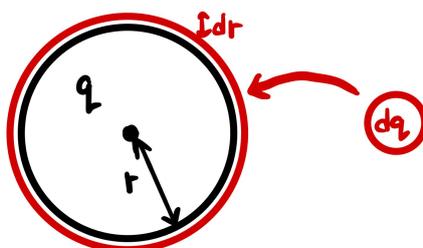
Just like in gravitation, brute force integration will become very involved for more complicated geometries. There are usually better ways to compute fields and potentials for continuous charge distributions.

1.2.2 "Building Up Layers"

The idea of "building up layers" is very important for potential energy. Let's consider the example below.

Example 1.2. Find the electric potential energy of a spherically symmetric charge distribution of total charge Q and radius R .

We can imagine constructing this spherically symmetric charge distribution by slowly adding on the layers of the sphere:



Let the volume charge density be $\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$. The infinitesimal potential energy between the new layer and the existing sphere is

$$dU = \frac{kq dq}{r} = \frac{k \left(\frac{4}{3}\rho\pi r^3 \right) d \left(\frac{4}{3}\rho\pi r^3 \right)}{r} = \frac{4}{3}k\pi\rho r^2 \left(4\pi\rho r^2 dr \right) = \frac{16}{3}k\pi^2\rho^2 r^4 dr$$

Hence, we can integrate from 0 to R to find the total potential energy:

$$U = \int dU = \int_0^R \frac{16}{3}k\pi^2\rho^2 r^4 dr = \frac{16}{3}k\pi^2\rho^2 \left(\frac{R^5}{5} \right) = \frac{3kQ^2}{5R}$$

1.3 Electric Dipoles

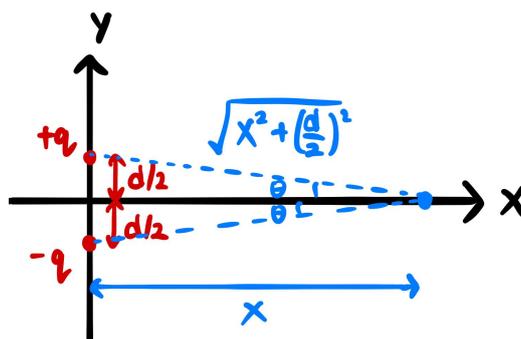
An electric dipole consists of a positive point charge $+q$ and a negative point charge $-q$ separated by a **small** distance d .

1.3.1 Electric Field of a Dipole

To calculate this, we treat the dipole as two point charges.

Example 1.3. Consider an electric dipole with a positive charge $+q$ located at $\left(0, \frac{d}{2}\right)$ and a negative charge $-q$ located at $\left(0, -\frac{d}{2}\right)$. (i) Find the electric field at $(x, 0)$. (ii) Find the electric field at $(0, y)$. Assume $x, y \gg d$.

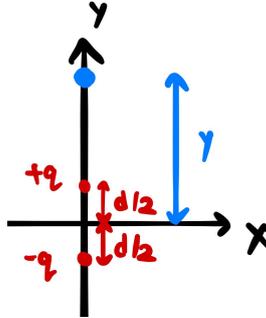
(i) The diagram of the set-up is as such:



Clearly, the x -component cancels out due to symmetry. Thus,

$$E_{net} = 2E_y = 2E \sin \theta = 2 \left(\frac{kq}{x^2 + \left(\frac{d}{2}\right)^2} \right) \left(\frac{\frac{d}{2}}{\sqrt{x^2 + \left(\frac{d}{2}\right)^2}} \right) = \frac{kqd}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}} \approx \frac{kqd}{x^3}$$

(ii) The diagram of the set-up is as such:



Clearly, there is no x -component. Thus, defining upwards as positive,

$$E_{net} = E_+ + E_- = \frac{kq}{\left(y - \frac{d}{2}\right)^2} - \frac{kq}{\left(y + \frac{d}{2}\right)^2} = kq \left(\frac{\left(y + \frac{d}{2}\right)^2 - \left(y - \frac{d}{2}\right)^2}{\left(y^2 - \left(\frac{d}{2}\right)^2\right)^2} \right) \approx kq \left(\frac{2yd}{y^4} \right) = \frac{2kqd}{y^3}$$

Interestingly, they "differ" by a factor of 2.

1.3.2 Electric Dipole Moment

Electric dipoles are often characterised by their **electric dipole moment, \mathbf{p}** , defined as

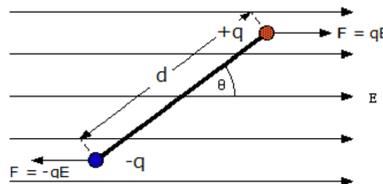
$$\mathbf{p} = q\mathbf{d} \quad (11)$$

where \mathbf{p} and \mathbf{d} are defined to point **from the negative to the positive charge**.

You often see the product qd pop up when dealing with dipoles (you already saw it in Example 1.3). The electric dipole moment is just a convenient quantity to simplify the product.

1.3.3 Potential Energy and Torque of an Electric Dipole

When an electric dipole is placed into an **external electric field, \mathbf{E}** , each charge experience a force. Due to the finite size of the dipole, this leads to a torque.



This torque is given by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \quad (12)$$

There is also a potential energy associated with this configuration, given by

$$U = -\mathbf{p} \cdot \mathbf{E} \quad (13)$$

1.4 Electric Flux and Gauss' Law

In gravitation, you have seen how to calculate gravitational flux and apply Gauss' Law. Given the similarities between electrostatics and gravitation, there similarly exists **electric flux**, Φ_E , and **Gauss' Law for Electricity**.

Electric flux is defined by:

$$\Phi_E = \oiint_A \mathbf{E} \cdot d\mathbf{A} \quad (14)$$

while Gauss' Law for Electricity states:

$$\oiint_A \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enclosed}}}{\epsilon_0} \quad (15)$$

Here, q_{enclosed} refers to the **total** charge enclosed by the Gaussian surface. **While the area integral looks daunting, in most cases symmetry allows you to just multiple field by area directly with minimal/zero integration!**

Remark. You need to be careful when calculating q_{enclosed} , as charges can be positive or negative (unlike masses, which are only positive)! Make sure to account for the correct signs.

From gravitation, you already know the different types of symmetries to spot when applying Gauss' Law. Let's go through them again in the context of electrostatics.

1.4.1 Spherical Symmetry

Example 1.4. Consider a sphere with uniform charge density ρ and radius R . Using Gauss' Law, find the electric field at a distance r away from it.

By symmetry the electric field must be radially outward and have the same magnitude everywhere on any concentric spherical surface, thus we choose a sphere of radius r concentric with the charge distribution to be our Gaussian surface.

Computing the charge enclosed,

$$q_{\text{enc}} = \rho \left(\frac{4}{3} \pi r^3 \right).$$

Applying Gauss' law,

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = E(r) (4\pi r^2) = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\rho \left(\frac{4}{3} \pi r^3 \right)}{\epsilon_0}.$$

Solving for $E(r)$,

$$E(r) = \frac{\rho r}{3\epsilon_0}, \quad 0 \leq r < R.$$

Hence the electric-field magnitude is *directly proportional to r* —i.e. linear—inside the sphere. The field points radially outward (or inward if $\rho < 0$) and satisfies:

$$E(0) = 0, \quad E(R) = \frac{\rho R}{3\epsilon_0}.$$

For $r \geq R$ all the charge $Q = \rho \frac{4}{3} \pi R^3$ is enclosed, so

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} = \frac{\rho R^3}{3\epsilon_0} \frac{1}{r^2},$$

which joins smoothly to the interior result at $r = R$.

1.4.2 Cylindrical Symmetry

Example 1.5. Consider an infinitely long charged wire of linear charge density λ . Using Gauss' Law, find the electric field at a distance x away from it.

Clearly, with the cylindrical symmetry around the wire, we draw a Gaussian cylinder of radius x and some length L around the wire. (Revisit gravitation if you aren't sure why.)

Thus, by Gauss' Law,

$$E(2\pi xL) = \frac{\lambda L}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 x} = \frac{2k\lambda}{x}$$

1.4.3 Planar Symmetry

Example 1.6. Consider an infinite charged plane with area charge density σ . Using Gauss' Law, find the electric field at a distance x away from it.

Clearly, with the planar symmetry around the plane, we draw a Gaussian "pillbox" of cross-sectional area A , with its faces at $\pm x$ parallel to the plane. (Revisit gravitation if you aren't sure why.)

Thus, by Gauss' Law,

$$E(2A) = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

1.4.4 Charged Boundaries

When we cross a boundary filled with charges, how does the electric field change right below and above the charged boundary?

Consider any random boundary filled with charge density σ . We can draw a Gaussian "pillbox", and by Gauss' Law,

$$E_{top}A - E_{bottom}A = \frac{\sigma A}{\epsilon_0} \quad (16)$$

Thus we have:

$$\Delta\mathbf{E}_\perp = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \quad (17)$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the plane.

This implies that electric field is **discontinuous** across charged boundaries!

A charged boundary also gives rise to **electrostatic pressure**, P_E , which is the electric force per unit area on the boundary:

$$P_E = \frac{\sigma^2}{2\epsilon_0} \quad (18)$$

which is very useful when calculating forces.

Interestingly, this may be a little unintuitive, but P_E is same as the **electric potential energy density** (by volume), u_E ! (You can perform a quick sanity check by dimensional analysis). u_E is defined as

$$u_E = \frac{U}{V} = \frac{1}{2}\epsilon_0 E^2 \quad (19)$$

Integrating over the volume occupied by the electric field gives the total electric potential energy:

$$U = \int u_E dV = \int \frac{1}{2}\epsilon_0 E^2 dV$$

1.5 Conductors

Conductors are a special type of material in which charges will redistribute themselves such that a lowest energy configuration is achieved. Most commonly, you would think of metals as conductors.

1.5.1 Properties of Conductors

There are a few general properties of conductors:

1. Inside a conductor, the net electric field $\mathbf{E} = \mathbf{0}$ (including external and induced fields).
2. The charge density $\rho = 0$ inside; excess charge resides only on the surface.
3. Just outside the conductor, \mathbf{E} is normal to the surface.
4. The conductor's surface is equipotential. Since $\mathbf{E} = \mathbf{0}$ inside, the potential V is uniform throughout.

These properties are sufficient to determine the charge distribution on conductors.

We can also discuss the electrostatic pressure here. Since we know that electric field inside the conductor is zero, the pressure can be expressed as such:

$$P_{E, \text{conductor}} = \sigma \left(\frac{E_{\text{out}} + 0}{2} \right) = \frac{\sigma E_{\text{out}}}{2} \quad (20)$$

1.5.2 Grounding

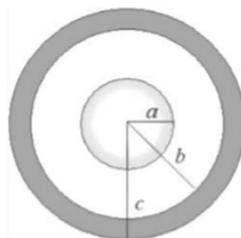
Grounding is the process of removing *excess* charges on an object by transferring electrons between it and the "ground" (an infinite reservoir of electrons).

In essence, grounding sets the **potential** of the object to be 0 (the reference).

Remark. It is **wrong** to assume that the total charge of the object is 0 after grounding! Grounding only removes *excess* charges to the point where the potential becomes 0.

In problems involving conductors/grounded conductors, you need to determine the charges on each **surface**. The example below illustrates.

Example 1.7 (Ricardo). Two concentric spherical conductors are described in the figure below. The smaller one is a solid sphere of radius a , and is charged with $+Q$. The larger one is a hollow sphere of inner radius b and outer radius c , and is charged with $-3Q$. (i) Find the functions $E(r)$ and $V(r)$. (ii) The larger sphere is now grounded. Find the functions $E(r)$ and $V(r)$. (iii) The larger sphere is now disconnected from the ground and is returned to having a charge $-3Q$. The smaller sphere is now grounded. Find the functions $E(r)$ and $V(r)$.



(i) Notice that the $-3Q$ on the larger sphere must spread out to have $-Q$ on the inner surface and $-2Q$ on the outer surface. This configuration ensures that when we consider the region

$b < r < c$, the enclosed charge is 0, hence the field is 0 inside the conductor.

Following which, this is just a simple application of Gauss' Law for spheres:

1. When $r < a$, there is no charge in the interior of the conductor, so $E = 0$.
2. When $a < r < b$, the enclosed charge is $q_{\text{enclosed}} = Q$, so $E = \frac{kQ}{r^2}$.
3. When $b < r < c$, we established that $E = 0$.
4. When $r > c$, the enclosed charge is $q_{\text{enclosed}} = -2Q$, so $E = -\frac{2kQ}{r^2}$.

Thus, the electric field is

$$E(r) = \begin{cases} 0, & r < a \\ \frac{kQ}{r^2}, & a < r < b \\ 0, & b < r < c \\ -\frac{2kQ}{r^2}, & r > c \end{cases}$$

We can begin integrating to find the potential, and you should obtain

$$V(r) = \begin{cases} -\frac{2kQ}{c} - kQ \left(\frac{1}{a} - \frac{1}{b} \right), & r < a \\ -\frac{2kQ}{c} - kQ \left(\frac{1}{r} - \frac{1}{b} \right), & a < r < b \\ -\frac{2kQ}{c}, & b < r < c \\ -\frac{2kQ}{r}, & r > c \end{cases}$$

The potential inside the conductor regions is constant, as you'd expect.

(ii) When the larger sphere is grounded, it has 0 potential. The only configuration that achieves this is if it has no charge (and hence the same potential as infinity, which has 0 potential).

Applying Gauss' Law, you should obtain

$$E(r) = \begin{cases} 0, & r < a \\ \frac{kQ}{r^2}, & a < r < b \\ 0, & b < r < c \\ 0, & r > c \end{cases}$$

We can begin integrating to find the potential, and you should obtain

$$V(r) = \begin{cases} kQ \left(\frac{1}{b} - \frac{1}{a} \right), & r < a \\ kQ \left(\frac{1}{b} - \frac{1}{r} \right), & a < r < b \\ 0, & r > b \end{cases}$$

(iii) It is very easy to fall into the trap and say that the charge on the smaller sphere is now 0. However, this is **wrong!** Clearly, the potential of the smaller sphere is not 0 in this case.

The correct method is to assume the smaller sphere, the inner surface of the larger sphere, and the outer surface of the larger sphere take on unknown charges q_1 , q_2 and q_3 respectively.

Since the larger sphere is not connected to the ground, conservation of charge must be obeyed:

$$q_2 + q_3 = -3Q$$

For there to be no field inside $b < r < c$, we must have:

$$q_1 = -q_2$$

Now, we can express the potential at $r = a$ in terms of these unknown charges:

$$V(a) = k \left(\frac{q_1}{a} + \frac{q_2}{b} + \frac{q_3}{c} \right) = 0 \quad \Rightarrow \quad q_1 \left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right) = \frac{3Q}{c}$$

With this, we can solve for all the unknown charges:

$$q_1 = \frac{3Q}{1 + c \left(\frac{1}{a} - \frac{1}{b} \right)}, \quad q_2 = -\frac{3Q}{1 + c \left(\frac{1}{a} - \frac{1}{b} \right)}, \quad q_3 = -\frac{3Qc \left(\frac{1}{a} - \frac{1}{b} \right)}{1 + c \left(\frac{1}{a} - \frac{1}{b} \right)}$$

Now, we can repeat a similar procedure to the other parts. Applying Gauss' Law, you should obtain

$$E(r) = \begin{cases} 0, & r < a \\ \frac{kq_1}{r^2}, & a < r < b \\ 0, & b < r < c \\ \frac{k(q_1 - 3Q)}{r^2}, & r > c \end{cases}$$

We can begin integrating to find the potential, and you should obtain

$$V(r) = \begin{cases} 0, & r > a \\ k \left(\frac{q_1}{a} - \frac{q_1}{b} + \frac{q_1 - 3Q}{c} \right), & a < r < b \\ k \left(\frac{q_1 - 3Q}{c} \right), & b < r < c \\ k \left(\frac{q_1 - 3Q}{r} \right), & r > c \end{cases}$$

1.6 Ideas

Many tricky electromagnetism problems involve the use of the following ideas.

1.6.1 Method of Images

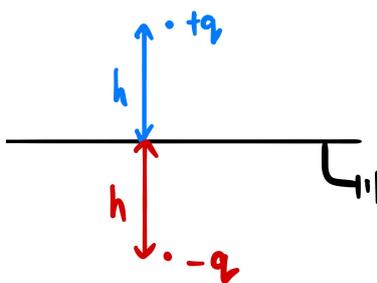
Often times, when dealing with charges and conductors, it may be difficult to calculate forces, fields and potentials. Let's take a look at a simple set-up below.

Example 1.8. Consider a charge $+q$ placed at a distance h above a thin, **grounded**, infinite conducting plane. (i) Find the force between the charge and the plane. (ii) Find the potential energy of this configuration.

(i) Naively, if you knew nothing about the method of images, you would perhaps to try find the induced charge distribution on the plane (setting the potential to be 0 everywhere on the conductor), and integrate across the plane. However, this is really tedious.

Instead, ask yourself: is there any other charge configuration that gives the same **boundary conditions**? (In this case, this refers to the potential of the whole plane being 0.)

Consider placing a charge $-q$ at a distance h below the plane.



Notice that by symmetry, *all* points on the plane have 0 potential (because the 2 charges cancel each other out)! This is the configuration that we are looking for.

With this, the force is simple:

$$F = \frac{kq^2}{(2h)^2} = \frac{kq^2}{4h^2}$$

(ii) Now that you know about this configuration, you may be tempted to just write

$$U = \frac{kq^2}{2h}$$

treating it as the potential energy between two point charges.

However, this is **wrong!** The reason is because the charge $-q$ is *not* a real charge. It is what we call an **image charge**, solely for the purpose of satisfying the boundary conditions.

The **correct answer** requires the definition of potential energy:

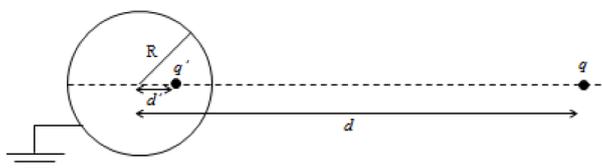
$$U = - \int_{\infty}^h F dh = - \int_{\infty}^h \frac{kq^2}{4h^2} dh = \frac{kq^2}{4h}$$

You might realise that the correct answer is exactly half of the wrong answer. The energy that we "over-counted" in the wrong answer comes from the interaction of the charges *on* the conducting plane (which are *real* charges!)

Remark. How are we so sure that this random configuration that we pulled out of nowhere is the *correct* configuration? The answer is the **uniqueness theorem**. Again, this is a case whereby you don't need to know how it works - you just need to know it exists and how to use it!

Other than the thin, grounded, infinite conducting plane, the other set-up you must recognise is the **hollow grounded sphere**.

Example 1.9 (SPhO 2015). Consider a charge q a distance d away from a grounded spherical conductor of radius R . The electric field and potential outside the sphere can be represented by an image charge q' a distance d' away from the centre of the sphere (there is no need to prove this). (i) Find q' and d' . (ii) Find the force F between the conductor and charge q , and determine whether it is attractive or repulsive.



(i) Again, the boundary condition is that the potential on the sphere is 0. Set up coordinates such that the charges lie along the x -axis, and the sphere is centred at the origin.

The potential at $(R, 0)$ and $(-R, 0)$ must be 0. Thus,

$$\frac{kq}{d-R} + \frac{kq'}{R-d'} = 0, \quad \frac{kq'}{R+d'} + \frac{kq}{d+R} = 0$$

These two simultaneous equations can be solved for q' and d' , to obtain

$$d' = \frac{R^2}{d}, \quad q' = -\frac{qR}{d}$$

If you know some geometry, you can see that the image charge q' and the real charge q are inversions of each other with respect to the sphere.

(ii) The force F can be found by considering the image charge:

$$F = \frac{kqq'}{(d-d')^2} = \frac{kq\left(-\frac{qR}{d}\right)}{\left(d-\frac{R^2}{d}\right)^2} = -\frac{kq^2R}{d\left(d-\frac{R^2}{d}\right)^2}$$

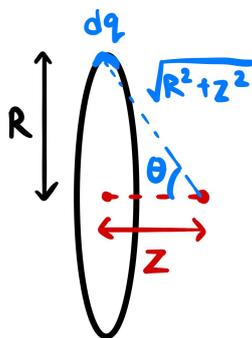
Clearly, the force is attractive as the charges have opposite signs.

1.6.2 Gauss' Law with $q_{\text{enclosed}} = 0$

So far, we have only applied Gauss' Law with q_{enclosed} being non-zero. It may also seem weird and useless to apply Gauss' Law with $q_{\text{enclosed}} = 0$. However, when applied correctly, it can speed up calculations! The following example illustrates.

Example 1.10 (IPhO 2021, modified). Consider a thin ring of linear charge density λ and radius R . (i) Find the electric field at a distance z away from the centre of the ring, **along its axis**. (ii) Find the electric field at a distance x away from the centre of the ring, **in its plane**. You may assume $x, z \ll R$.

(i) The first part is simple and is more easily done by naive integration.

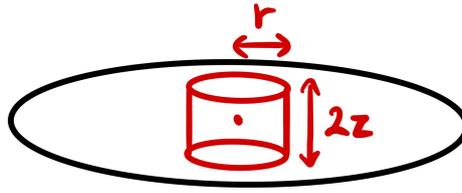


Only the z -component survives upon integration, hence

$$E = \int dE \cos \theta = \int \frac{k dq}{z^2 + R^2} \frac{z}{\sqrt{z^2 + R^2}} = \frac{kqz}{(z^2 + R^2)^{\frac{3}{2}}} = \frac{2\pi k \lambda R z}{(z^2 + R^2)^{\frac{3}{2}}} \approx \frac{2\pi k \lambda z}{R^2}$$

Not bad for around half a point on the IPhO!

(ii) This is where we use the Gauss' Law trick, which requires the result of (i). Consider a small Gaussian cylinder of radius r and height $2z$ constructed at the centre of the ring:



Since $q_{\text{enclosed}} = 0$, by Gauss' Law,

$$\Phi_{E, \text{total}} = \oiint_{\text{top}} \mathbf{E} \cdot d\mathbf{A} + \oiint_{\text{bottom}} \mathbf{E} \cdot d\mathbf{A} + \oiint_{\text{side}} \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enclosed}}}{\epsilon_0} = 0$$

Because the cylinder is small, we can evaluate the flux through each of the faces by assuming a constant field across the surfaces. Thus,

$$\oiint_{\text{top}} \mathbf{E} \cdot d\mathbf{A} = \oiint_{\text{bottom}} \mathbf{E} \cdot d\mathbf{A} = \pi r^2 \left(\frac{2\pi k \lambda z}{R^2} \right) = \frac{2\pi^2 k \lambda z r^2}{R^2}$$

Thus,

$$\oiint_{\text{side}} \mathbf{E} \cdot d\mathbf{A} = - \left(\oiint_{\text{top}} \mathbf{E} \cdot d\mathbf{A} + \oiint_{\text{bottom}} \mathbf{E} \cdot d\mathbf{A} \right) = - \frac{4\pi^2 k \lambda z r^2}{R^2} = E (2\pi r (2z))$$

Thus, the electric field along the axis of the ring is

$$E = - \frac{\pi k \lambda r}{R^2}$$

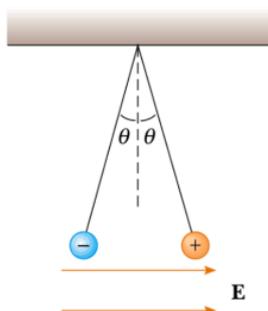
where the negative sign implies that the field points radially inwards.

By using this trick, we didn't even have to integrate at all!

2 Problems

Problems are arranged in roughly increasing difficulty.

Problem 2.1 (SPhO 2008). Two small spheres, each of mass 2.00 g, are suspended by light strings 10.0 cm in length. A uniform electric field is applied in the x -direction. The spheres have charges equal to -5.00×10^{-8} C and $+5.00 \times 10^{-8}$ C. Determine the electric field that enables the spheres to be in equilibrium at an angle 10.0° .



Problem 2.2 (SPhO 2015). A particle of mass m and charge q , travelling with initial velocity v from far away, undergoes a head-on "collision" with an identical particle initially at rest. (i) Find the distance of closest approach. (ii) Find the velocities at closest approach. (iii) Find the final velocities.

Problem 2.3. (Griffiths 2.18) In this problem you will derive some important results.

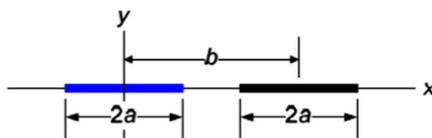
(a) Consider a sphere of radius R and uniform charge density ρ . Find the electric field everywhere.

(b) Now two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, are placed so that they partially overlap. Call the vector from the positive center to the negative center \mathbf{d} . Find the electric field in the overlap region.

(c) Now instead of a sphere, consider two infinitely long, uniformly charged cylinders that are partially overlapped and carrying uniform charge densities ρ and $-\rho$. Again, call the vector from the positive center to the negative center \mathbf{d} . Find the electric field within the overlap region.

Problem 2.4. A soap film is made of a conductive liquid with uniform surface charge σ . Given the surrounding pressure is P_o , what is the pressure within the film? Neglect surface tension.

Problem 2.5. Two identical thin rods of length $2a$ carry equal charges $+Q$ uniformly distributed along their lengths. The rods lie along the x -axis with their centres separated by a distance $b > 2a$.



(a) Show that the magnitude of the force exerted by the left rod on the right one is

$$F = \frac{Q^2}{16\pi\epsilon_0 a^2} \ln\left(\frac{b^2}{b^2 - 4a^2}\right).$$

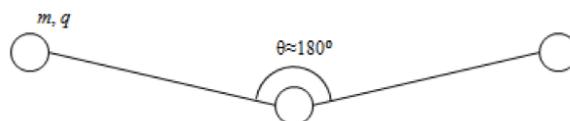
(b) Explain qualitatively how this force would be different if each rod were a conductor.

Problem 2.6. A point charge e is placed at a distance R from the centre of a metallic sphere of radius a , with $R > a$. The sphere is conducting, but insulated from the surroundings and is electrically neutral.

- Determine the potential on the surface of the sphere.
- Determine the resultant force acting on the charge.

Problem 2.7. A circular ring of radius R_1 carries a uniformly distributed charge Q . It is placed near a neutral sphere of radius R_2 , with their centres separated by a distance L . The plane of the ring is perpendicular to the straight line connecting the centres of the ring and the sphere. Determine the average electric potential on the surface of the sphere, \bar{U} .

Problem 2.8 (SPhO 2015). 3 identical coins of mass m and charge q are connected by 2 strings of length d . Given that the period of oscillation is T , find q .



Problem 2.9. A cube of length l possesses a uniform volume charge density ρ . Find the ratio of the electric potential at one of its vertices to that at its centre. *Hint: use scaling arguments.*

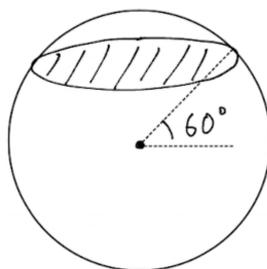
Problem 2.10. (Ricardo) A hole of radius R is carved out of a thin infinite plane with a positive surface charge density σ that is uniform. A point charge q of mass m is placed at the center of the hole. Neglecting all gravitational effects:

- Show that the center of the hole corresponds to an equilibrium position for the charge.
- Determine the stability of this equilibrium when the charge is displaced slightly in the direction normal to the plane. Discuss how the stability depends on the sign and magnitude of q .
- If the equilibrium is stable, find the angular frequency of small oscillations of the charge. You may find the result of the previous problem useful.

Problem 2.11. A point charge q is placed a distance $a/2$ above the centre of a square of surface charge density σ and side length a . Find the force exerted by the square on the point charge. *Hint: the value $\frac{a}{2}$ was chosen for a special reason.*

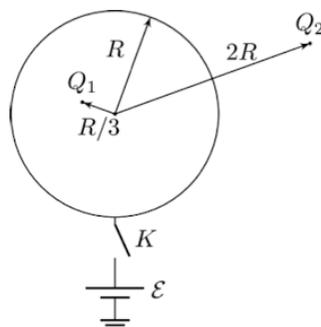
Problem 2.12. For a solid sphere of radius R and uniform charge density ρ , calculate:

- The electric field within the sphere.
- The electric flux through the circular area (see diagram below).



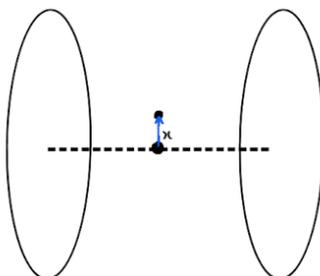
Problem 2.13. A particle of charge q is moved from infinity to the centre of a hollow conducting spherical shell of inner radius R and thickness t , through a very tiny hole in the shell. Determine the work required.

Problem 2.14. An initially uncharged conducting spherical shell of radius R is connected to ground through a battery with EMF ε and an open switch K . A point charge Q_1 is placed inside the shell at a distance $R/3$ from its centre, and a point charge Q_2 is placed outside the shell at a distance $2R$ from its centre.



- (a) Determine the potential at the centre of the shell when the switch K is open.
 (b) The switch K is then closed and equilibrium is reached. Determine the charge on the shell.

Problem 2.15. Two circular plates of radius R are placed a distance d apart. They are charged with surface charge densities $-\sigma$. A small particle of charge $+q$ and mass m is placed at the midpoint of the line connecting the centres of the two plates. It is displaced slightly in a direction perpendicular to this line. Determine the period of small oscillations it performs.



3 Advanced Problems

Problem 3.1. This problem involves a cool idea that has appeared elsewhere before. There are two point charges, $q_1 > 0$ and $q_2 < 0$, in empty space. An electric field line leaves q_1 at an angle α from the line connecting the two charges. Determine whether this field line hits q_2 , and if so, at what angle β from the line connecting the two charges. (Hint: this can be done without solving any differential equations.)

Problem 3.2 (200 Puzzling Physics Problems). Consider a uniformly charged spherical shell of radius R and total charge Q . (i) Find the net electrostatic force that the Southern hemisphere exerts on the Northern hemisphere. (ii) Generalise part (i) to the case where the sphere is split into two parts by a plane whose minimum distance to the sphere's centre is h . (iii) Generalise part (i) to the case of two hemispherical shells with uniform charge density, opposite orientation, and the same centre, but have different total charges q and Q and different radii r and R respectively, where $r < R$.

Problem 3.3. A thin-walled dielectric hemisphere, negatively charged with surface density $-\sigma$, is placed on a horizontal table. A point-like ball of mass m is carefully placed at its top. Determine the minimum positive charge Q of the ball such that it remains in a state of stable equilibrium at the top of the hemisphere. The charges on the hemisphere and the ball are not redistributed. The free-fall acceleration is g .